

The quantum supergroup $\mathrm{SPO}_q(2n|2m)$ and an $\mathrm{SPO}_q(2n|2m)$ -covariant quantum Weyl superalgebra

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Abstract

Recently, the R -matrix of the symplecto-orthogonal quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$ in the vector representation has been calculated. In the present work, this R -matrix is used to introduce the corresponding quantum supergroup $\mathrm{SPO}_q(2n|2m)$ and to construct an $\mathrm{SPO}_q(2n|2m)$ -covariant quantum Weyl superalgebra.

1 Introduction

Once the quantum algebras and quantum groups had been introduced [1], [2], [3], it was obvious to apply the same techniques also in the super case. Indeed, at present there exist a considerable number of papers on quantum supergroups. Most of them consider deformations of the general linear supergroup $GL(m|n)$ [4], [5], but more recently there also appeared a paper by Lee and Zhang [6], in which a deformation of $OSP(1|2n)$ is discussed, and a paper by Zhang [7], dealing with a deformation of $OSP(2|2n)$.

Quantum supergroups and quantum superalgebras are dual objects. This implies that there are (at least) two methods to introduce a quantum supergroup. One may regard the quantum supergroup as the primary object and define it by means of the R -matrix approach, thus generalizing the methods of Ref. [3] to the super case. On the other hand, one may also start from the quantum superalgebra and construct the (algebra of functions on the) quantum supergroup as a sub-Hopf-superalgebra of the finite dual of the former. In Manin's paper [4] the first approach is used, while Zhang [5], [6], [7] prefers to use the second. It is believed that both methods should lead to the same objects, however, since it is notoriously difficult to put the duality of quantum (super)groups and quantum (super)algebras on a firm basis (including a proof that the dual pairing is non-degenerate(!)), there is still a lot to be done to substantiate this belief (for example, see Ref. [8]; this paper contains an extensive list of references on quantum supergroups and quantum superalgebras).

The present paper is a direct sequel to Ref. [9], in which I have calculated the R -matrix of the quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$ in the vector representation. Here, I am going to use this R -matrix to introduce the corresponding quantum supergroup $SPO_q(2n|2m)$ and to construct some of its comodule superalgebras, among which is an $SPO_q(2n|2m)$ -covariant quantum Weyl superalgebra. This will be done by means of techniques which (in the non-super case) have been described in Ref. [3] (i.e., we use the first of the two approaches described above). However, as we are going to see, in the super case some amendments are necessary in order to cope with the notorious problem that the representations are not necessarily completely reducible. The reason for considering the $SPO_q(2n|2m)$ quantum supergroup rather than $OSP_q(2m|2n)$ has been explained in Ref. [9].

The present work falls into two parts. The first half (Sections 2 to 4) contains results which are applicable to arbitrary quantum supergroups (not just the symplecto-orthogonal ones). In fact, these results are formulated in the general graded framework, where Γ is an arbitrary abelian group and σ is a commutation factor on Γ . In the second half (Sections 5 and 6), the special case of the $SPO_q(2n|2m)$ quantum supergroups is considered.

More precisely, this paper is set up as follows. In Section 2 we are going to recall some elementary facts about matrix elements of graded representations of a σ -bialgebra or σ -Hopf algebra. These facts will be used to motivate the definitions

and constructions of the subsequent sections. In particular, they serve to find the appropriate sign factors (i.e., commutation factors) by which the formulae of the present paper differ from those in the non-graded setting.

Usually, a quantum (super)group is constructed in two steps. First, one introduces the corresponding bi-(super)algebra $A(R)$, then one amends this definition to obtain the quantum (super)group itself, which is a Hopf (super)algebra. Basic to the construction of $A(R)$ is a finite-dimensional (graded) vector space V and some (even) linear operator R in $V \otimes V$ which, in the more advanced part of the theory, is assumed to be invertible and to satisfy the (graded) Yang–Baxter equation. The σ -bialgebras $A(R)$ are discussed to some extent in Section 3. In particular, we prove (under the foregoing additional assumptions on R) that $A(R)$ is coquasitriangular.

Section 4 is devoted to the discussion of graded comodules and comodule algebras over $A(R)$. We concentrate on those of these structures that are derived from the vector comodule or its dual.

In Section 5 we introduce the quantum supergroup $\mathrm{SPO}_q(2n|2m)$ and derive some of its properties. In particular, we show that $\mathrm{SPO}_q(2n|2m)$ is coquasitriangular, and that there exists a natural dual pairing of the Hopf superalgebras $\mathrm{SPO}_q(2n|2m)$ and $U_q(\mathfrak{spo}(2n|2m))$. (According to the general folklore, both of these results are expected to be true.)

Section 6 contains a discussion of some graded $\mathrm{SPO}_q(2n|2m)$ -comodule superalgebras. After describing the general setting, we concentrate on the construction of the $\mathrm{SPO}_q(2n|2m)$ -covariant quantum Weyl superalgebras.

The final Section 7 contains a discussion of our results and some hints to further research.

We close this introduction by some comments on our conventions and notation. Throughout the present work, the base field will be an arbitrary field \mathbb{K} of characteristic zero. The pairing of a vector space and its dual will be denoted by pointed brackets, and the action of an algebra A on an A -module will be denoted by a dot. It should be obvious that we are going to work in the general setting of graded algebraic structures (see Ref. [10]). To remind the reader of this fact, the graded tensor product of linear maps and of graded algebras will be denoted by $\overline{\otimes}$. However, it might be worthwhile to stress that even if the overbar seems to be missing this does *not* indicate that we have returned to the non-graded setting.

2 Some elementary properties of matrix elements of representations

In the present section we recall some elementary properties of the matrix elements of a graded representation of a σ -Hopf algebra (or σ -bialgebra). These properties will serve to motivate the definitions and constructions given later.

First of all, we introduce some notation. Throughout the present section, Γ will be an abelian group, and σ will be a commutation factor on Γ with values in \mathbb{K} . All gradations appearing in this section are understood to be Γ -gradations. In the following, we shall freely use the notation and results of Ref. [10].

Let V be a *finite-dimensional* graded vector space, let $V^{*\text{gr}}$ be the graded dual of V , and let $\text{Lgr}(V)$ be the space of all linear maps of V into itself that can be written as a sum of homogeneous linear maps. In Ref. [10], we have denoted the latter space by $\text{Lgr}(V, V)$. Since V is finite-dimensional, it is known that, regarded as vector spaces, $V^{*\text{gr}}$ is equal to V^* , the dual of V , and $\text{Lgr}(V)$ is equal to the space of all linear maps of V into itself. We use the present notation in order to stress that $V^{*\text{gr}}$ is a graded vector space, and that $\text{Lgr}(V)$ is a graded associative algebra.

Suppose now that $(e_i)_{i \in I}$ is a homogeneous basis of V , where I is some index set, and that e_i is homogeneous of degree $\eta_i \in \Gamma$. Let $(e'_i)_{i \in I}$ be the basis of $V^{*\text{gr}}$ that is dual to $(e_i)_{i \in I}$, and let $(E_{ij})_{i, j \in I}$ be the basis of $\text{Lgr}(V)$ that canonically corresponds to $(e_i)_{i \in I}$:

$$E_{ij}(e_k) = \delta_{jk} e_i \quad \text{for all } i, j, k \in I.$$

Then e'_i is homogeneous of degree $-\eta_i$, and E_{ij} is homogeneous of degree $\eta_i - \eta_j$.

It is well-known that the class of Γ -graded vector spaces, endowed with the usual tensor product of graded vector spaces and with the twist maps defined by means of σ , forms a tensor category (see Ref. [10] for details). A (generalized) Hopf algebra H living in this category is called a σ -Hopf algebra. More explicitly, H is a Γ -graded associative algebra with a unit element, and it is endowed with a coproduct Δ , a counit ε , and an antipode S , which satisfy the obvious axioms (in the category). In particular, this implies that the structure maps Δ , ε , and S are homogeneous of degree zero. Needless to say, σ -bialgebras are introduced similarly. (For generalized Hopf algebras living in more general categories, see Ref. [11].)

Let H be a σ -Hopf algebra, and let H° be the set of all linear forms on H that vanish on a graded two-sided ideal of H of finite codimension. This definition generalizes the definition given in Sweedler's book [12] to the general graded case considered here.

It is easy to see that the properties derived in Ref. [12] can immediately be extended to the present setting. In particular, this is true for the criteria characterizing the elements of H° . We don't want to go into detail here, but mention some properties not given there.

First of all we note that in the preceding definition the graded two-sided ideals can be replaced by graded left or right ideals (indeed, every graded left or right ideal of finite codimension contains a graded two-sided ideal of finite codimension). Secondly, the elements of H° can also be characterized by the following property:

A linear form f on H belongs to H° if and only if there exists a finite-dimensional graded (left) H -module V , an element $x \in V$, and a linear form $x' \in V^{*\text{gr}}$ such that

$$f(h) = \langle x', h \cdot x \rangle \quad \text{for all } h \in H.$$

This property shows very clearly that H° is closely related to the theory of finite-dimensional graded representations of H .

Obviously, H° is a graded subspace of $H^{*\text{gr}}$, the graded dual of H . Actually, much more is true, for the structure of H as a σ -Hopf algebra leads to a similar structure on H° . More precisely, the transpose of Δ induces the multiplication in H° , the counit ε is the unit element of H° , the transpose of the product map of H induces the coproduct Δ_\circ of H° , the evaluation at the unit element of H is the counit ε_\circ of H° , and the transpose of S induces the antipode S_\circ of H° . This σ -Hopf algebra H° is called the finite (or continuous) dual of H .

Using the notation introduced at the beginning of this section, let us now assume in addition that V is a graded H -module, and that $\pi : H \rightarrow \text{Lgr}(V)$ is the graded representation afforded by it. If $i, j \in I$, we define the linear form π_{ij} on H by

$$\pi_{ij}(h) = \langle e'_i, \pi(h)e_j \rangle \quad \text{for all } h \in H.$$

Obviously, π_{ij} is an element of H° , and is homogeneous of degree $\eta_j - \eta_i$. Moreover, we have

$$\pi(h) = \sum_{i,j \in I} \pi_{ij}(h) E_{ij} \quad \text{for all } h \in H.$$

Then our first important observation is that

$$\Delta_\circ(\pi_{ij}) = \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) \pi_{ik} \otimes \pi_{kj} \quad (2.1)$$

for all $i, j \in I$. By definition of the antipode, this implies that

$$\sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) S_\circ(\pi_{ik}) \pi_{kj} = \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) \pi_{ik} S_\circ(\pi_{kj}) = \delta_{i,j} 1_{H^\circ}, \quad (2.2)$$

for all $i, j \in I$, where we have used the obvious fact that

$$\varepsilon_\circ(\pi_{ij}) = \delta_{ij} \quad \text{for all } i, j \in I.$$

Let us next recall that V has a canonical structure of a graded right H° -comodule. It is uniquely fixed by the requirement that it induces, in a canonical way, the original structure of a graded left H -module on V . More precisely, let

$$\delta : V \longrightarrow V \otimes H^\circ$$

be the structure map of V , regarded as a graded right H° -comodule, let $x \in V$ be homogeneous of degree ξ , and set

$$\delta(x) = \sum_r x_r^0 \otimes x_r^1,$$

with $x_r^0 \in V$ and $x_r^1 \in H^\circ$. Then, if $h \in H$ is homogeneous of degree η , we have

$$h \cdot x = \pi(h)x = \sigma(\eta, \xi) \sum_r x_r^0 \langle x_r^1, h \rangle ,$$

where \langle , \rangle denotes the dual pairing of H° and H . Explicitly, we find

$$\delta(e_i) = \sum_{j \in I} \sigma(\eta_i, \eta_j - \eta_i) e_j \otimes \pi_{ji} , \quad (2.3)$$

for all $i \in I$.

Similarly, let V' be the graded *right* H -module that is dual to V . Regarded as a graded vector space, V' is equal to $V^{*\text{gr}}$, and the module structure is given by

$$\langle x' \cdot h, x \rangle = \langle x', h \cdot x \rangle ,$$

for all $h \in H$, $x \in V$, and $x' \in V'$. (Of course, this time \langle , \rangle denotes the dual pairing of $V^{*\text{gr}}$ and V .) More explicitly, we have

$$e'_i \cdot h = \sum_{j \in I} \pi_{ij}(h) e'_j ,$$

for all $h \in H$ and all $i \in I$. Then V' has a canonical structure of a graded left H° -comodule. It is uniquely fixed by the requirement that it induces, in a canonical way, the original structure of a graded right H -module on V . More precisely, let

$$\delta' : V' \longrightarrow H^\circ \otimes V'$$

be the structure map of V' , regarded as a graded left H° -comodule, let $z \in V'$, and set

$$\delta'(z) = \sum_r z_r^{-1} \otimes z_r^0 ,$$

where $z_r^{-1} \in H^\circ$, and where $z_r^0 \in V'$ is homogeneous of degree ζ_r^0 . Then, if $h \in H$ is homogeneous of degree η , we have

$$z \cdot h = \sum_r \sigma(\zeta_r^0, \eta) \langle z_r^{-1}, h \rangle z_r^0 .$$

Explicitly, we find

$$\delta'(e'_i) = \sum_{j \in I} \sigma(\eta_j, \eta_i - \eta_j) \pi_{ij} \otimes e'_j , \quad (2.4)$$

for all $i \in I$.

Let us next consider two cases where our assumptions are more restrictive. In the first case, we suppose that on V there exists a non-degenerate invariant bilinear form b which is homogeneous of degree zero (see Appendix A of Ref. [9] for some comments on invariant bilinear forms). Define a linear map

$$f_r : V \longrightarrow V^{*\text{gr}}$$

by

$$\langle f_r(y), x \rangle = \sigma(\eta, \xi) b(x, y) ,$$

where x and y are elements of V which are homogeneous of degrees ξ and η , respectively. Then the assumption that b is invariant is equivalent to the requirement that

$$\pi(S(h)) = f_r^{-1} \circ (\pi(h))^T \circ f_r \quad \text{for all } h \in H ,$$

where T denotes the σ -transpose of a linear map (see Eqn. (A.5) of Ref. [9]). In terms of matrix elements, this condition takes the form

$$\pi_{ij}(S(h)) = \sum_{k, \ell} (F_r^{-1})_{ik} (\pi(h)^T)_{k\ell} (F_r)_{\ell j} \quad \text{for all } i, j \in I \text{ and all } h \in H , \quad (2.5)$$

where F_r is the matrix of f_r :

$$f_r(e_j) = \sum_{i \in I} (F_r)_{ij} e'_i \quad \text{for all } j \in I .$$

To understand the meaning of Eqn. (2.5) we note that

$$\pi_{ij} \circ S = S_\circ(\pi_{ij}) , \quad (2.6)$$

moreover, we recall that the matrix elements of $\pi(h)^T$ are given by

$$(\pi(h)^T)_{k\ell} = \sigma(\eta_\ell, \eta_\ell - \eta_k) \pi_{\ell k}(h) . \quad (2.7)$$

Thus Eqn. (2.5) expresses $S_\circ(\pi_{ij})$ as a linear combination of the matrix elements $\pi_{\ell k}$.

For convenience, we also give a formula for the matrix F_r . Setting

$$C_{ij} = b(e_i, e_j)$$

and

$$\sigma_{ij} = \sigma(\eta_i, \eta_j) \quad , \quad \sigma_i = \sigma(\eta_i, \eta_i) ,$$

for all $i, j \in I$, we obtain

$$(F_r)_{ij} = \sigma_{ji} C_{ij} = \sigma_i C_{ij} = C_{ij} \sigma_j \quad \text{for all } i, j \in I . \quad (2.8)$$

In the second case, we assume in addition that H is quasitriangular, i.e., that we are given a universal R -matrix \mathcal{R} for H . By definition, \mathcal{R} is an invertible element of $H \otimes H$ and is homogeneous of degree zero. Moreover, it satisfies the following relations:

$$\mathcal{R} \Delta(h) = (P \Delta(h)) \mathcal{R} \quad \text{for all } h \in H , \quad (2.9)$$

$$(\Delta \otimes \text{id}_H)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23} \quad , \quad (\text{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12} . \quad (2.10)$$

Our notation is standard: $P : H \overline{\otimes} H \longrightarrow H \overline{\otimes} H$ is the twist map (in the graded sense), and the elements \mathcal{R}_{12} , \mathcal{R}_{13} , and \mathcal{R}_{23} of $H \overline{\otimes} H \overline{\otimes} H$ are defined by

$$\begin{aligned}\mathcal{R}_{12} &= \mathcal{R} \otimes 1 \quad , \quad \mathcal{R}_{23} = 1 \otimes \mathcal{R} \\ \mathcal{R}_{13} &= (P \otimes \text{id}_H)(\mathcal{R}_{23}) = (\text{id}_H \otimes P)(\mathcal{R}_{12}) .\end{aligned}$$

Let us define the R -matrix R (more precisely, the R -operator R) acting in $V \otimes V$ by

$$R = (\pi \otimes \pi)(\mathcal{R}) .$$

Using the notation

$$\mathcal{R} = \sum_r R_r^1 \otimes R_r^2 , \tag{2.11}$$

we have

$$R = \sum_{ijkl \in I} R_{ij,kl} E_{ik} \overline{\otimes} E_{j\ell} ,$$

with

$$R_{ij,kl} = \sum_r \pi_{ik}(R_r^1) \pi_{j\ell}(R_r^2) .$$

Acting with $\pi \otimes \pi$ on both sides of Eqn. (2.9), we obtain

$$\sum_{a,b \in I} \sigma(\eta_j - \eta_\ell, \eta_a - \eta_k) R_{ij,ab} \pi_{ak} \pi_{b\ell} = \sum_{a,b \in I} \sigma(\eta_j - \eta_b, \eta_a - \eta_k) \pi_{jb} \pi_{ia} R_{ab,kl} , \tag{2.12}$$

for all $i, j, k, \ell \in I$. These equations generalize the famous RTT-relations. By using the following technical device, they can be written in the familiar form.

Consider the algebra

$$\text{Lgr}(V) \overline{\otimes} \text{Lgr}(V) \overline{\otimes} H^\circ \simeq \text{Lgr}(V \otimes V) \overline{\otimes} H^\circ$$

and define the following elements therein:

$$\pi_1 = \sum_{i,j \in I} E_{ij} \otimes \text{id}_V \otimes \pi_{ij} \quad , \quad \pi_2 = \sum_{i,j \in I} \text{id}_V \otimes E_{ij} \otimes \pi_{ij} . \tag{2.13}$$

Then the Eqns. (2.12) take the form

$$(R \otimes \varepsilon) \pi_1 \pi_2 = \pi_2 \pi_1 (R \otimes \varepsilon) \tag{2.14}$$

(recall that ε is the unit element of H°).

Remark 2.1. Some comments on the notation are in order. If A and B are two graded associative algebras, we denote their graded tensor product by $A \overline{\otimes} B$ and the decomposable tensors in $A \overline{\otimes} B$ by $a \otimes b$ (where $a \in A$ and $b \in B$, see the

appendix of Ref. [13]). Let us next recall that there exists a canonical isomorphism of graded algebras

$$\mathrm{Lgr}(V) \overline{\otimes} \mathrm{Lgr}(V) \longrightarrow \mathrm{Lgr}(V \otimes V) ;$$

for all $g, g' \in \mathrm{Lgr}(V)$, it maps the tensor $g \otimes g'$ onto the graded tensor product $g \overline{\otimes} g'$ of the linear maps g and g' (see the Eqns. (3.4) and (3.5) of Ref. [10]). Originally, R is an element of $\mathrm{Lgr}(V \otimes V)$. However, occasionally (as above) it is useful to regard it also as an element of $\mathrm{Lgr}(V) \overline{\otimes} \mathrm{Lgr}(V)$. As such, it will be written in the form $R = \sum_{ijk\ell \in I} R_{ij,k\ell} E_{ik} \otimes E_{j\ell}$. The reader should be careful not to confound $E_{ik} \otimes E_{j\ell}$ with the usual (non-graded) tensor product of the linear maps E_{ik} and $E_{j\ell}$ (which has been used in Ref. [9], but will never be used in the present work).

Now let A be a graded sub- σ -bialgebra of H° . Then the universal R -matrix yields a bilinear form ϱ on A that is defined by

$$\varrho(a, b) = \langle a \otimes b, \mathcal{R} \rangle \quad \text{for all } a, b \in A .$$

Explicitly, we have

$$\varrho(a, b) = \sum_r \sigma(\rho_r^1, \rho_r^2) \langle a, R_r^1 \rangle \langle b, R_r^2 \rangle \quad \text{for all } a, b \in A .$$

(We use the notation introduced in Eqn. (2.11) and assume that R_r^1 and R_r^2 are homogeneous of degrees ρ_r^1 and ρ_r^2 , respectively.) In particular, if the linear forms π_{ij} belong to A , we have

$$\varrho(\pi_{ik}, \pi_{j\ell}) = \sigma(\eta_i - \eta_k, \eta_j - \eta_\ell) R_{ij,k\ell} \quad \text{for all } i, j, k, \ell \in I . \quad (2.15)$$

Moreover, the defining properties of a universal R -matrix imply the following properties of ϱ . First of all, the bilinear form ϱ is homogeneous of degree zero. Secondly, ϱ is convolution invertible.

To understand this statement, we recall that $A \otimes A$ has a canonical structure of a graded coalgebra. Correspondingly, its graded dual $(A \otimes A)^{\mathrm{gr}}$ is a graded algebra (whose multiplication is called convolution). Since homogeneous bilinear forms on A can be canonically identified with homogeneous linear forms on $A \otimes A$, it makes sense to require that ϱ is invertible in $(A \otimes A)^{\mathrm{gr}}$, and this is the meaning of the statement above. More explicitly, this is to say that there exists a (necessarily unique) bilinear form ϱ' on A , which is homogeneous of degree zero and satisfies

$$\sum_{rs} \sigma(\alpha_r^2, \beta_s^1) \varrho(a_r^1, b_s^1) \varrho'(a_r^2, b_s^2) = \sum_{rs} \sigma(\alpha_r^2, \beta_s^1) \varrho'(a_r^1, b_s^1) \varrho(a_r^2, b_s^2) = \varepsilon(a) \varepsilon(b) \quad (2.16)$$

for all $a, b \in A$. In these equations, we have set

$$\Delta(a) = \sum_r a_r^1 \otimes a_r^2 \quad , \quad \Delta(b) = \sum_s b_s^1 \otimes b_s^2 ,$$

and the elements $a_r^1, a_r^2, b_s^1, b_s^2 \in A$ are supposed to be homogeneous of the degrees $\alpha_r^1, \alpha_r^2, \beta_s^1, \beta_s^2$, respectively.

Using the same notation, the relation (2.9) implies that

$$\sum_{rs} \sigma(\alpha_r^2, \beta_s^1) \varrho(a_r^1, b_s^1) a_r^2 b_s^2 = \sum_{rs} \sigma(\alpha_r^1 + \alpha_r^2, \beta_s^1) b_s^1 a_r^1 \varrho(a_r^2, b_s^2) \quad (2.17)$$

for all $a, b \in A$, and the relations (2.10) imply that

$$\varrho(aa', b) = \sum_s \sigma(\alpha', \beta_s^1) \varrho(a, b_s^1) \varrho(a', b_s^2) \quad (2.18)$$

$$\varrho(a, bb') = \sum_r \varrho(a_r^1, b') \varrho(a_r^2, b) \quad (2.19)$$

for all $a, a', b, b' \in A$, where a' is supposed to be homogeneous of degree α' .

A bilinear form ϱ on a σ -bialgebra A that has the properties given above is called a *universal r -form* for A , and a σ -bialgebra (or σ -Hopf algebra) endowed with a universal r -form is said to be *coquasitriangular*.

If ϱ is a universal r -form for a bialgebra A , then so is $\varrho' \circ P$, where $P : A \overline{\otimes} A \longrightarrow A \overline{\otimes} A$ denotes the twist. The conditions above imply that

$$\varrho(a, 1) = \varrho(1, a) = \varepsilon(a) \quad \text{for all } a \in A. \quad (2.20)$$

(To prove this one has to use the fact that ϱ is invertible.) Moreover, if A is a σ -Hopf algebra, it follows that

$$\varrho(S(a), b) = \varrho'(a, b) \quad , \quad \varrho'(a, S(b)) = \varrho(a, b) \quad , \quad (2.21)$$

for all $a, b \in A$.

Let A^{cop} (resp. A^{aop}) be the σ -bialgebra which, regarded as a graded algebra (resp. as a graded coalgebra) is equal to A , but whose coalgebra (resp. algebra) structure is opposite (in the graded sense) to that of A . Then the Eqns. (2.18), (2.19), and (2.20) show that ϱ is a dual pairing (possibly degenerate) of the σ -bialgebras A^{cop} and A , and also of the σ -bialgebras A and A^{aop} .

3 The σ -bialgebras $A(R)$

We keep the notation introduced at the beginning of Section 2 and start by recalling a simple fact (see Ref. [12] for the non-graded case). Let C be a graded coalgebra. Equivalently, this means that C is a graded vector space and a coalgebra, and that the structure maps of the latter are homogeneous of degree zero. Consider the tensor algebra $T(C)$ of the graded vector space C . It is well-known that $T(C)$ is a $\mathbb{Z} \times \Gamma$ -graded algebra. Then there exists a unique coalgebra structure on $T(C)$

that converts $T(C)$ into a σ -bialgebra, and such that the canonical injection $\iota : C \rightarrow T(C)$ is a homomorphism of graded coalgebras. The σ -bialgebra $T(C)$ and the injection ι have the following universal property:

If B is any σ -bialgebra, and if $\varphi : C \rightarrow B$ is a homomorphism of graded coalgebras, there exists a unique σ -bialgebra homomorphism $\bar{\varphi} : T(C) \rightarrow B$ such that $\varphi = \bar{\varphi} \circ \iota$.

We apply the foregoing construction to the following special case. Let the index set I and the family of degrees $\eta = (\eta_i)_{i \in I}$ be as at the beginning of Section 2. We consider the following graded coalgebra $C(\eta)$: It has a basis $(X_{ij})_{i,j \in I}$, such that X_{ij} is homogeneous of degree $\eta_j - \eta_i$, and such that the coproduct Δ and the counit ε are given by

$$\begin{aligned} \Delta(X_{ij}) &= \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) X_{ik} \otimes X_{kj} \\ \varepsilon(X_{ij}) &= \delta_{ij} , \end{aligned}$$

for all $i, j \in I$. Obviously, according to Eqn. (2.1), the graded coalgebra $C(\eta)$ is (isomorphic to) the dual of the graded algebra $\text{Lgr}(V)$. We stress that $T(C(\eta))$ is a free algebra with free generators X_{ij} ; $i, j \in I$.

Now let

$$R : V \otimes V \longrightarrow V \otimes V$$

be a linear map which is *homogeneous of degree zero*. As before, we use the notation

$$R = \sum_{ijk\ell \in I} R_{ij,k\ell} E_{ik} \bar{\otimes} E_{j\ell} . \quad (3.1)$$

The homogeneity condition is equivalent to the requirement that

$$R_{ij,k\ell} = 0 \quad \text{if } \eta_i + \eta_j \neq \eta_k + \eta_\ell .$$

Consider the graded algebra

$$\text{Lgr}(V) \bar{\otimes} \text{Lgr}(V) \bar{\otimes} T(C(\eta)) \simeq \text{Lgr}(V \otimes V) \bar{\otimes} T(C(\eta))$$

and introduce the following elements therein:

$$X_1 = \sum_{i,j \in I} E_{ij} \otimes \text{id}_V \otimes X_{ij} \quad , \quad X_2 = \sum_{i,j \in I} \text{id}_V \otimes E_{ij} \otimes X_{ij} \quad (3.2)$$

(see the Eqns. (2.13)). Define the elements $X_{ij,k\ell} \in T(C(\eta))$; $i, j, k, \ell \in I$, through the following equation:

$$(R \otimes 1)X_1X_2 - X_2X_1(R \otimes 1) = \sum_{ijk\ell \in I} E_{ik} \otimes E_{j\ell} \otimes X_{ij,k\ell}$$

(see Eqn. (2.14)). According to the definition, $X_{ij,k\ell}$ is homogeneous of degree $\eta_k - \eta_i + \eta_\ell - \eta_j$ with respect to the Γ -gradation and homogeneous of degree 2 with respect to the \mathbb{Z} -gradation. Explicitly, we obtain

$$X_{ij,k\ell} = \sum_{a,b \in I} \sigma(\eta_j - \eta_\ell, \eta_a - \eta_k) R_{ij,ab} X_{ak} X_{b\ell} - \sum_{a,b \in I} \sigma(\eta_j - \eta_b, \eta_a - \eta_k) X_{jb} X_{ia} R_{ab,k\ell} ,$$

for all $i, j, k, \ell \in I$.

Let $J(R)$ be the (two-sided) ideal of $T(C(\eta))$ that is generated by the elements $X_{ij,k\ell}$; $i, j, k, \ell \in I$. It is not difficult to see that $J(R)$ is, in fact, a Γ -graded biideal and a $\mathbb{Z} \times \Gamma$ -graded ideal of $T(C(\eta))$. Consequently, the quotient

$$A(R) = T(C(\eta))/J(R)$$

inherits from $T(C(\eta))$ the structure of a σ -bialgebra and also of a $\mathbb{Z} \times \Gamma$ -graded algebra. (Our notation is somewhat incomplete since, obviously, $A(R)$ also depends on σ and on $\eta = (\eta_i)_{i \in I}$.) The canonical image of X_{ij} in $A(R)$ will be denoted by t_{ij} , for all $i, j \in I$.

Consider the graded algebra

$$\text{Lgr}(V) \overline{\otimes} \text{Lgr}(V) \overline{\otimes} A(R) \simeq \text{Lgr}(V \otimes V) \overline{\otimes} A(R)$$

and introduce the following elements therein:

$$T_1 = \sum_{i,j \in I} E_{ij} \otimes \text{id}_V \otimes t_{ij} \quad , \quad T_2 = \sum_{i,j \in I} \text{id}_V \otimes E_{ij} \otimes t_{ij} . \quad (3.3)$$

Then $A(R)$ is the universal graded algebra, generated by elements t_{ij} ; $i, j \in I$, which are homogeneous of degree $\eta_j - \eta_i$ and satisfy the following RTT-relation

$$(R \otimes 1)T_1T_2 = T_2T_1(R \otimes 1) . \quad (3.4)$$

Explicitly, this relation is equivalent to

$$\sum_{a,b \in I} \sigma(\eta_j - \eta_\ell, \eta_a - \eta_k) R_{ij,ab} t_{ak} t_{b\ell} = \sum_{a,b \in I} \sigma(\eta_j - \eta_b, \eta_a - \eta_k) t_{jb} t_{ia} R_{ab,k\ell} , \quad (3.5)$$

for all $i, j, k, \ell \in I$. Moreover, the coproduct Δ and the counit ε are uniquely fixed by the equations

$$\Delta(t_{ij}) = \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) t_{ik} \otimes t_{kj} \quad (3.6)$$

$$\varepsilon(t_{ij}) = \delta_{ij} , \quad (3.7)$$

for all $i, j \in I$.

It is well-known that there exists a useful equivalent formulation of these relations. Let $P : V \otimes V \rightarrow V \otimes V$ be the twist. Explicitly, we obtain

$$P = \sum_{i,j \in I} \sigma(\eta_i, \eta_j) E_{ji} \overline{\otimes} E_{ij}$$

(the factor $\sigma(\eta_i, \eta_j)$ is not a misprint). If g and g' are two elements of $\text{Lgr}(V)$ which are homogeneous of degrees γ and γ' , respectively, we have

$$P \circ (g \overline{\otimes} g') = \sigma(\gamma, \gamma') (g' \overline{\otimes} g) \circ P.$$

Define

$$\hat{R} = PR.$$

Then the relations (3.4) can also be written in the form

$$(\hat{R} \otimes 1) T_1 T_2 = T_1 T_2 (\hat{R} \otimes 1). \quad (3.8)$$

Up to now, R could be an arbitrary element of $\text{Lgr}(V \otimes V)$ that is homogeneous of degree zero. This indicates that we haven't yet proved anything really substantial about the σ -bialgebras $A(R)$. Presumably, the most important property of the $A(R)$'s is that, under certain natural additional assumptions, they are coquasitriangular. More precisely, we have the following theorem.

Theorem 1. We keep the notation used above. Let R be an element of $\text{Lgr}(V \otimes V)$ that is homogeneous of degree zero. Suppose in addition that R is invertible and satisfies the graded Yang-Baxter equation. Then there exists a unique universal r -form ϱ on $A(R)$ such that

$$\varrho(t_{ik}, t_{j\ell}) = \sigma(\eta_i - \eta_k, \eta_j - \eta_\ell) R_{ij, k\ell} \quad \text{for all } i, j, k, \ell \in I. \quad (3.9)$$

(We are using the notation introduced in Eqn. (3.1).)

Sketch of the proof

We start from the following fact which, in the non-graded case, has been mentioned in Ref. [14]. Consider two graded coalgebras C and D and the corresponding σ -bialgebras $T(C)$ and $T(D)$ introduced at the beginning of this section. If $\psi_1 : D \times C \rightarrow \mathbb{K}$ is any bilinear form which is homogeneous of degree zero, there exists a unique dual pairing ψ of the σ -bialgebras $T(D)$ and $T(C)$ that extends ψ_1 .

Applying this result to $C = C(\eta)$ and $D = C(\eta)^{\text{op}}$, we obtain a unique dual pairing ψ of the σ -bialgebras $T(C(\eta))^{\text{cop}} = T(C(\eta)^{\text{op}})$ and $T(C(\eta))$ such that

$$\psi(X_{ik}, X_{j\ell}) = \sigma(\eta_i - \eta_k, \eta_j - \eta_\ell) R_{ij, k\ell} \quad \text{for all } i, j, k, \ell \in I.$$

Let us next investigate under which conditions this form induces a bilinear form on $A(R)$. For this to be the case it is necessary that

$$\psi(X_{ij, k\ell}, X_{ab}) = 0 \quad \text{and} \quad \psi(X_{ab}, X_{ij, k\ell}) = 0$$

for all $i, j, k, \ell, a, b \in I$. Each of these conditions is satisfied if and only if the graded Yang–Baxter equation holds for R . Conversely, if this is the case, the bilinear form ψ indeed induces a bilinear form ϱ on $A(R)$. Obviously, ϱ is a dual pairing of the σ –bialgebras $A(R)^{\text{cop}}$ and $A(R)$.

Our next task is to show that the generalized commutation relations (2.17) are satisfied. The basic first step of the proof consists in checking that these relations hold for $a = t_{ik}$ and $b = t_{j\ell}$, for all $i, j, k, \ell \in I$. But up to unimportant overall factors, these are just the defining relations for $A(R)$.

Finally, we have to prove that ϱ is convolution invertible. If ϱ' is an inverse of ϱ , the Eqns. (2.16), applied to $a = t_{ik}$ and $b = t_{j\ell}$, show that R is invertible: In fact, the inverse of R is given in terms of ϱ' exactly as R is given in terms of ϱ (see Eqn. (3.9)). Accordingly, we now assume that R is invertible.

It is well-known (and easy to see) that if R satisfies the graded Yang–Baxter equation, then so does $\tilde{R} = PR^{-1}P$. Moreover, \tilde{R} and R define the same σ –bialgebra: We have

$$A(\tilde{R}) = A(R) . \quad (3.10)$$

Consequently, we can apply the foregoing results with R replaced by \tilde{R} , which gives another dual pairing $\tilde{\varrho}$ of the σ –bialgebras $A(R)^{\text{cop}}$ and $A(R)$. According to the preceding discussion, we expect that $\varrho' = \tilde{\varrho} \circ P$ is the inverse of ϱ , and this is indeed the case.

4 Graded comodules and comodule algebras over $A(R)$

We keep the notation introduced at the beginning of Section 2 and in Section 3. In the present section we are going to discuss certain natural graded $A(R)$ –comodules and $A(R)$ –comodule algebras. Actually, we make a slight generalization. The point is the following. At last, we are not so much interested in the σ –bialgebras $A(R)$ themselves, but rather in certain σ –Hopf algebra envelopes therefrom. Typically, the latter are obtained from $A(R)$ by adjoining the inverses of certain elements of $A(R)$, or by requiring additional relations (i.e., by going to a quotient). In any case, such an envelope will be a σ –bialgebra A , and there will be a natural homomorphism of σ –bialgebras $A(R) \rightarrow A$. For convenience, the natural image of t_{ij} in A will also be denoted by t_{ij} . This will be the setting considered in the present section. Stated differently, A will be a σ –bialgebra containing certain elements t_{ij} ; $i, j \in I$, such that t_{ij} is homogeneous of degree $\eta_j - \eta_i$, for all $i, j \in I$, such that the relation (3.4) (or its equivalents (3.5), (3.8)) is satisfied, and such that the Eqns. (3.6) and (3.7) hold true.

To begin with, we note that V is a graded right A –comodule in a natural way: The structure map

$$\delta_1 : V \longrightarrow V \otimes A$$

is defined by

$$\delta_1(e_i) = \sum_{j \in I} \sigma(\eta_i, \eta_j - \eta_i) e_j \otimes t_{ji} , \quad (4.1)$$

for all $i \in I$ (see Eqn. (2.3)).

Similarly, $V^{*\text{gr}}$ is a graded left A -comodule: The structure map

$$\delta'_1 : V^{*\text{gr}} \longrightarrow A \otimes V^{*\text{gr}}$$

is defined by

$$\delta'_1(e'_i) = \sum_{j \in I} \sigma(\eta_j, \eta_i - \eta_j) t_{ij} \otimes e'_j , \quad (4.2)$$

for all $i \in I$. We note that the graded comodule $V^{*\text{gr}}$ is dual to the graded comodule V in the sense that

$$(\text{id}_A \otimes \langle \rangle) \circ (\delta'_1 \otimes \text{id}_V) = (\langle \rangle \otimes \text{id}_A) \circ (\text{id}_{V^{*\text{gr}}} \otimes \delta_1) . \quad (4.3)$$

(These are maps of $V^{*\text{gr}} \otimes V$ into A , and $\langle \rangle : V^{*\text{gr}} \otimes V \rightarrow \mathbb{K}$ is given by the dual pairing.) Obviously, if one of the structure maps δ_1 and δ'_1 is known, the other is uniquely fixed by this equation.

Remark 4.1. At this point we insert a remark that could have been made earlier. Looking at the Eqns. (4.1) and (4.2), we are tempted to introduce the elements

$$\tilde{t}_{ij} = \sigma(\eta_j, \eta_i - \eta_j) t_{ij} \quad \text{for all } i, j \in I . \quad (4.4)$$

Then these equations take the familiar form

$$\delta_1(e_i) = \sum_{j \in I} e_j \otimes \tilde{t}_{ji} \quad , \quad \delta'_1(e'_i) = \sum_{j \in I} \tilde{t}_{ij} \otimes e'_j$$

for all $i \in I$, which implies that

$$\Delta(\tilde{t}_{ij}) = \sum_{k \in I} \tilde{t}_{ik} \otimes \tilde{t}_{kj}$$

for all $i, j \in I$. (Needless to say, this equation can easily be checked directly.) Summarizing, the elements \tilde{t}_{ij} seem to have properties which are closer to what we are accustomed to from the non-graded case. However, unfortunately I do not know how to write the commutation relations for the \tilde{t}_{ij} in a factorized form like Eqn. (3.4). My conclusion is that, depending on the problem at hand, both types of generators may have advantages. (Actually, in the proof of Theorem 1 I have mostly used the generators \tilde{t}_{ij} , whereas for the proof of Eqn. (3.10) the factorization in Eqn. (3.4) is of great help.)

Once the graded A -comodule structures of V and $V^{*\text{gr}}$ are defined, they can be used to convert $T(V)$ and $T(V^{*\text{gr}})$ into graded A -comodule algebras. Thus, there exists a unique homomorphism of graded algebras

$$\delta : T(V) \longrightarrow T(V) \overline{\otimes} A$$

such that Eqn. (4.1) (with δ_1 replaced by δ) is satisfied, and it is easy to see that this map converts $T(V)$ into a graded right A -comodule, hence into a graded right A -comodule algebra. Note that, for every integer $n \geq 0$, the subspace $T_n(V) = V^{\otimes n}$ is a graded subcomodule of $T(V)$, in fact, this is just the n th tensorial power of the graded comodule V . Its structure map is the map

$$\delta_n : T_n(V) \longrightarrow T_n(V) \otimes A \quad (4.5)$$

induced by δ . (The map δ_1 is the map introduced in Eqn. (4.1).)

Similarly, there exists a unique homomorphism of graded algebras

$$\delta' : T(V^{*\text{gr}}) \longrightarrow A \overline{\otimes} T(V^{*\text{gr}})$$

such that Eqn. (4.2) (with δ'_1 replaced by δ') is satisfied, and it is easy to see that this map converts $T(V^{*\text{gr}})$ into a graded left A -comodule algebra. For every integer $n \geq 0$, the subspace $T_n(V^{*\text{gr}}) = (V^{*\text{gr}})^{\otimes n}$ is a graded subcomodule of $T(V^{*\text{gr}})$, in fact, this is just the n th tensorial power of the graded comodule $V^{*\text{gr}}$. Its structure map is the map

$$\delta'_n : T_n(V^{*\text{gr}}) \longrightarrow A \otimes T_n(V^{*\text{gr}}) \quad (4.6)$$

induced by δ' . (The map δ'_1 is the map introduced in Eqn. (4.2).)

The graded comodule $T_n(V^{*\text{gr}})$ is dual to the graded comodule $T_n(V)$. Indeed, since the case $n = 0$ is trivial, we may assume that $n \geq 1$. Let us define a canonical pairing

$$\langle \cdot, \cdot \rangle : T_n(V^{*\text{gr}}) \times T_n(V) \longrightarrow \mathbb{K}$$

by

$$\langle x'_1 \otimes \dots \otimes x'_n, x_1 \otimes \dots \otimes x_n \rangle = \prod_{1 \leq a < b \leq n} \sigma(\xi'_b, \xi_a) \prod_{1 \leq c \leq n} \langle x'_c, x_c \rangle$$

for all $x_1, \dots, x_n \in V$; $x'_1, \dots, x'_n \in V^{*\text{gr}}$, which are homogeneous of the degrees $\xi_1, \dots, \xi_n, \xi'_1, \dots, \xi'_n$, respectively. Then δ_n and δ'_n are related by an equation analogous to Eqn. (4.3).

The algebra $T(V)$ is free, with free generators e_i ; $i \in I$. In order to obtain more interesting examples of graded A -comodule algebras, we have to construct suitable quotients of $T(V)$. In principle, this is easily done (see Ref. [15]). Let W be any graded A -subcomodule of $T(V)$, let $J(W)$ be the ideal of $T(V)$ that is generated by W , and let

$$B_r(W) = T(V)/J(W)$$

be the corresponding quotient. Of course, $B_r(W)$ is a graded algebra. By assumption, we have

$$\delta(W) \subset W \otimes A .$$

Obviously, this implies that

$$\delta(J(W)) \subset J(W) \otimes A .$$

Consequently, the map δ induces a map

$$\delta : B_r(W) \longrightarrow B_r(W) \overline{\otimes} A$$

(note the abuse of notation), and it is clear that this map converts $B_r(W)$ into a graded right A -comodule algebra. Let us denote the canonical image of e_i in $B_r(W)$ by x_i , for all $i \in I$. Then $B_r(W)$ is the universal graded algebra, generated by elements x_i ; $i \in I$, which are homogeneous of degree η_i and satisfy the relations “given by W ”. Moreover, the comodule structure of $B_r(W)$ is fixed by

$$\delta(x_i) = \sum_{j \in I} \sigma(\eta_i, \eta_j - \eta_i) x_j \otimes t_{ji} \quad \text{for all } i \in I . \quad (4.7)$$

Instead of writing the defining relations in general, let us give a simple example. Suppose that W is a graded subcomodule of $T_2(V) = V \otimes V$. Let $(a^s)_{1 \leq s \leq r}$ be a family of elements of W that generates the vector space W , and let us write

$$a^s = \sum_{i,j \in I} a_{ij}^s e_i \otimes e_j \quad \text{for } 1 \leq s \leq r .$$

Then the defining relations of $B_r(W)$ can be written in the form

$$\sum_{i,j \in I} a_{ij}^s x_i x_j = 0 \quad \text{for } 1 \leq s \leq r .$$

In practice, it might be difficult to prove that a graded subspace of $T(V)$ is a graded subcomodule. Indeed, all we know about the algebra structure of A are the commutation relations (3.5). It is at this point where the techniques of Ref. [3] are of great help.

Let $S \in \text{Lgr}(V \otimes V)$ be homogeneous of degree zero. As usual, we write

$$S = \sum_{ijkl \in I} S_{ij,kl} E_{ik} \overline{\otimes} E_{j\ell} . \quad (4.8)$$

We recall that the homogeneity condition is equivalent to the requirement that

$$S_{ij,kl} = 0 \quad \text{if } \eta_i + \eta_j \neq \eta_k + \eta_\ell .$$

Let us also introduce the elements

$$T_1, T_2 \in \text{Lgr}(V) \overline{\otimes} \text{Lgr}(V) \overline{\otimes} A \simeq \text{Lgr}(V \otimes V) \overline{\otimes} A$$

as in Eqn. (3.3). Then S is an endomorphism of the graded A -comodule $V \otimes V$ if and only if

$$(S \otimes 1)T_1T_2 = T_1T_2(S \otimes 1) . \quad (4.9)$$

Explicitly, this relation is equivalent to

$$\sum_{a,b \in I} \sigma(\eta_j - \eta_\ell, \eta_a - \eta_k) S_{ij,ab} t_{ak} t_{b\ell} = \sum_{a,b \in I} \sigma(\eta_j - \eta_b, \eta_i - \eta_k) t_{ia} t_{jb} S_{ab,k\ell} , \quad (4.10)$$

for all $i, j, k, \ell \in I$.

A look at Eqn. (3.8) shows that $S = \hat{R}$ satisfies Eqn. (4.9). Indeed, the algebra $A(R)$ is *defined* such that \hat{R} is an endomorphism of the graded $A(R)$ -comodule $V \otimes V$. It follows that every polynomial in \hat{R} satisfies this equation as well. Since the image and the kernel of every such S are graded A -subcomodules of $V \otimes V$ (and hence of $T(V)$), we have found some natural candidates for the subcomodule W .

We can now comment on how the present formulation generalizes the approach of Ref. [3]. In that reference, the $A(R)$ -comodule $V \otimes V$ is completely reducible (provided that q is generic), and the subcomodules can be constructed as the images of suitable polynomials in \hat{R} . As we are going to see, for the quantum supergroups $\text{SPO}_q(2n|2n)$ this is not true anymore, and we are forced to use the generalized method described above.

The graded left A -comodule algebras can be constructed in a completely analogous way, and hence it should not be necessary to give all the details. In particular, for every graded subcomodule W' of $T(V^{*\text{gr}})$, we obtain a graded left A -comodule algebra $B_\ell(W')$. However, one point should be mentioned. In this second case, we need to know the endomorphisms of the graded A -comodule $V^{*\text{gr}} \otimes V^{*\text{gr}}$. Every element of $\text{Lgr}(V^{*\text{gr}} \otimes V^{*\text{gr}})$ can be written as the graded transpose S^T of some element $S \in \text{Lgr}(V \otimes V)$, and S^T is an endomorphism of the graded A -comodule $V^{*\text{gr}} \otimes V^{*\text{gr}}$ if and only if S is an endomorphism of the graded A -comodule $V \otimes V$, i.e., if and only if the equivalent conditions (4.9) and (4.10) are satisfied. Using the notation (4.8) and assuming that S is homogeneous of degree zero, we have

$$S^T(e'_i \otimes e'_j) = \sum_{k,\ell \in I} \sigma(\eta_j, \eta_k - \eta_i) S_{ij,k\ell} e'_k \otimes e'_\ell .$$

5 The quantum supergroup $\text{SPO}_q(2n|2m)$

In the present and in the subsequent section we are going to apply the general theory to the R -matrix that has been calculated in Ref. [9], namely, to the R -matrix of

$U_q(\mathfrak{spo}(2n|2m))$ in the vector representation. Correspondingly, we shall use the notation introduced in that reference. First of all, this implies that in the following we choose

$$\Gamma = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

and define

$$\sigma(\alpha, \beta) = (-1)^{\alpha\beta} \quad \text{for all } \alpha, \beta \in \mathbb{Z}_2 .$$

Moreover, we set

$$r = n + m \quad , \quad d = n - m ,$$

and the index set I is equal to

$$I = \{-r, -r+1, \dots, -2, -1, 1, 2, \dots, r-1, r\} .$$

The degrees η_i ; $i \in I$, are given by

$$\eta_i = \begin{cases} \bar{0} & \text{for } 1 \leq |i| \leq n \\ \bar{1} & \text{for } n+1 \leq |i| \leq r . \end{cases}$$

We recall that

$$\sigma_i = \sigma(\eta_i, \eta_i) \quad \text{and} \quad \sigma_{i,j} = \sigma(\eta_i, \eta_j) \quad \text{for all } i, j \in I .$$

In the present setting, V is the vector module of $U_q(\mathfrak{spo}(2n|2m))$. It carries a $U_q(\mathfrak{spo}(2n|2m))$ -invariant bilinear form b^q , whose matrix $C^q = (C_{i,j}^q)$ with respect to the basis $(e_i)_{i \in I}$ has been given at the end of Section 4 of Ref. [9]. Finally, we note that the R -matrix R and the braid generator \hat{R} are given by

$$\begin{aligned} R &= \sum_i q^{\sigma_i} E_{i,i} \bar{\otimes} E_{i,i} + \sum_i q^{-\sigma_i} E_{i,i} \bar{\otimes} E_{-i,-i} \\ &\quad + \sum_{i \neq j, -j} E_{i,i} \bar{\otimes} E_{j,j} \\ &\quad + (q - q^{-1}) \sum_{i < j} \sigma_i E_{j,i} \bar{\otimes} E_{i,j} \\ &\quad - (q - q^{-1}) \sum_{i < j} \sigma_i \sigma_j \sigma_{i,j} ((C^q)^{-1})_{-j,j} C_{i,-i}^q E_{j,i} \bar{\otimes} E_{-j,-i} \\ \hat{R} &= \sum_i \sigma_i q^{\sigma_i} E_{i,i} \bar{\otimes} E_{i,i} + \sum_i \sigma_i q^{-\sigma_i} E_{-i,i} \bar{\otimes} E_{i,-i} \\ &\quad + \sum_{i \neq j, -j} \sigma_i E_{j,i} \bar{\otimes} E_{i,j} \\ &\quad + (q - q^{-1}) \sum_{i < j} E_{i,i} \bar{\otimes} E_{j,j} \\ &\quad - (q - q^{-1}) \sum_{i < j} \sigma_i \sigma_{i,j} ((C^q)^{-1})_{-j,j} C_{i,-i}^q E_{-j,i} \bar{\otimes} E_{j,-i} , \end{aligned}$$

and that the operator K in $V \otimes V$ is equal to

$$K = \sum_{i,j,k,\ell} \sigma_{k,j} \sigma_{k,\ell} ((C^q)^{-1})_{i,j} C_{k,\ell}^q E_{i,k} \bar{\otimes} E_{j,\ell} .$$

Remark 5.1. In Ref. [9] we have assumed that the base field \mathbb{K} is equal to \mathbb{C} (since this assumption has been made by Yamane in Ref. [16]). Obviously, the R -matrix obtained in Ref. [9] makes sense under the present more general assumptions, and the properties derived there are still true. Actually, also the definition of the quantum superalgebra $U_q(\mathfrak{spo}(2n|2m))$, its properties, and the discussion of the vector module and of its tensorial square all remain valid. (Indeed, according to my understanding Yamane's entire paper is correct with \mathbb{C} replaced by \mathbb{K} .) For convenience, we still assume that q is not a root of unity, although this condition could be relaxed.

The σ -bialgebras $A(R)$ don't have an antipode (except for the degenerate case where I is empty): This follows immediately by means of the \mathbb{Z} -gradation of these algebras. Hence some amendments are necessary if we want to obtain a σ -Hopf algebra. In view of the comments made in Section 2, it is easy to guess how this can be achieved under the present assumptions.

Let T^{st} be the super-transpose of the matrix T , defined by

$$(T^{\text{st}})_{k,\ell} = \sigma(\eta_\ell, \eta_\ell - \eta_k) t_{\ell,k} \quad \text{for all } k, \ell \in I ,$$

(see Eqn. (2.7)), and let F_r be the $I \times I$ -matrix over \mathbb{K} , defined by

$$(F_r)_{i,j} = \sigma_{j,i} C_{i,j}^q = \sigma_i C_{i,j}^q = C_{i,j}^q \sigma_j \quad \text{for all } i, j \in I$$

(see Eqn. (2.8)). Then the Eqns. (2.2), (2.5), and (2.6) suggest to define the $I \times I$ -matrix (with elements in $A(R)$)

$$T' = F_r^{-1} T^{\text{st}} F_r ,$$

explicitly:

$$t'_{i,j} = \sigma_i \sigma_{i,j} ((C^q)^{-1})_{i,-i} t_{-j,-i} C_{-j,j}^q ,$$

and to investigate, for all $i, j \in I$, the expressions

$$\begin{aligned} Q'_{i,j} &= \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) t'_{i,k} t_{k,j} \\ Q''_{i,j} &= \sum_{k \in I} \sigma(\eta_k - \eta_i, \eta_k - \eta_j) t_{i,k} t'_{k,j} . \end{aligned}$$

This can be carried out as in Ref. [17]. The basic observation is that the operator K can be written as a polynomial in \hat{R} (see Eqn. (7.3) of Ref. [9]). This implies that the relations (4.10) are satisfied for $S = K$, and it is easy to evaluate these relations.

If $i \neq -j$ and $k \neq -\ell$, both sides of these relations are trivially equal to zero, and no information can be obtained.

If $i = -j$ and $k \neq -\ell$, it follows that

$$\sum_{a,b \in I} \sigma_{a,\ell} \sigma_{b,a} C_{a,b}^q t_{a,k} t_{b,\ell} = 0 \quad \text{for } k \neq -\ell .$$

If $i \neq -j$ and $k = -\ell$, we find that

$$\sum_{a,b \in I} \sigma_{i,b} ((C^q)^{-1})_{a,b} t_{i,a} t_{j,b} = 0 \quad \text{for } i \neq -j .$$

If $i = -j$ and $k = -\ell$, we set

$$\begin{aligned} Q' &= (C_{k,-k}^q)^{-1} \sum_{a,b \in I} \sigma_{a,k} \sigma_{b,a} C_{a,b}^q t_{a,k} t_{b,-k} \\ Q'' &= (((C^q)^{-1})_{i,-i})^{-1} \sigma_i \sum_{a,b \in I} \sigma_{i,b} ((C^q)^{-1})_{a,b} t_{i,a} t_{-i,b} . \end{aligned}$$

Then the relation under investigation says that

$$Q' = Q'' .$$

Obviously, Q' does not depend on i, j , and Q'' does not depend on k, ℓ . This implies that Q' and Q'' do not depend on i, j, k, ℓ , and the equations above can be summarized as follows:

$$\sum_{a,b \in I} \sigma_{\ell,b} \sigma_{b,a} C_{a,b}^q t_{a,k} t_{b,\ell} = C_{k,\ell}^q Q' \tag{5.1}$$

$$\sum_{a,b \in I} \sigma_{i,b} ((C^q)^{-1})_{a,b} t_{i,a} t_{j,b} = \sigma_{i,j} ((C^q)^{-1})_{i,j} Q'' . \tag{5.2}$$

Remark 5.2. Using the generators $\tilde{t}_{i,j}$ as defined in Eqn. (4.4), i.e.,

$$\tilde{t}_{i,j} = \sigma_j \sigma_{j,i} t_{i,j} \quad \text{for all } i, j \in I ,$$

and defining the super-transpose \tilde{T}^{st} of the $I \times I$ -matrix $\tilde{T} = (\tilde{t}_{k,\ell})$ by

$$(\tilde{T}^{\text{st}})_{k,\ell} = \sigma_\ell \sigma_{k,\ell} \tilde{t}_{\ell,k} \quad \text{for all } k, \ell \in I$$

(see Eqn. (2.7)), it is easy to see that the relations (5.1) and (5.2) can be written in the form

$$\tilde{T}^{\text{st}} C^q \tilde{T} = Q' C^q \quad , \quad \tilde{T} (C^q)^{-1} \tilde{T}^{\text{st}} = Q'' (C^q)^{-1} .$$

Now it is easy to see that

$$Q'_{i,j} = \delta_{i,j} Q' \quad , \quad Q''_{i,j} = \delta_{i,j} Q'' . \tag{5.3}$$

In the following, we set

$$Q' = Q'' = Q .$$

Obviously, the element Q is homogeneous of degree zero. Using the Eqns. (5.1) and (5.2) once again, we can also show that Q is group-like, i.e., that

$$\Delta(Q) = Q \otimes Q \quad , \quad \varepsilon(Q) = 1 .$$

Finally, we prove that Q lies in the center of $A(R)$. Consider the following elements of the graded tensor product $\text{Lgr}(V) \otimes A(R)$:

$$T_0 = \sum_{i,j \in I} E_{i,j} \otimes t_{i,j} \quad , \quad T'_0 = \sum_{i,j \in I} E_{i,j} \otimes t'_{i,j} .$$

Then the Eqns. (5.3) can be written in the following form:

$$T'_0 T_0 = T_0 T'_0 = \mathbb{I} \otimes Q \tag{5.4}$$

(where we have set $\mathbb{I} = \text{id}_V$). These equations imply that

$$T_0(\mathbb{I} \otimes Q) = T_0 T'_0 T_0 = (\mathbb{I} \otimes Q) T_0 .$$

Since Q is homogeneous of degree zero, this is equivalent to

$$t_{i,j} Q = Q t_{i,j} \quad \text{for all } i, j \in I ,$$

which proves our claim.

After these preparations, we can define the quantum supergroup $\text{SPO}_q(2n|2m)$, as follows. Let $J(Q)$ be the ideal of $A(R)$ that is generated by $Q - 1$. Since Q is homogeneous of degree zero and group-like, $J(Q)$ is a graded biideal of $A(R)$. Consequently, the quotient

$$\text{SPO}_q(2n|2m) = A(R)/J(Q) \tag{5.5}$$

inherits from $A(R)$ the structure of a bi-superalgebra (i.e., a σ -bialgebra with respect to the commutation factor σ of supersymmetry). Let

$$\omega : A(R) \longrightarrow \text{SPO}_q(2n|2m)$$

be the canonical map. Until further notice, we are going to use the notation

$$\omega(t_{i,j}) = \bar{t}_{i,j} \quad \text{for all } i, j \in I .$$

Similarly, we introduce the elements $\bar{t}'_{i,j}$ and the $I \times I$ -matrices \bar{T} and \bar{T}' .

Let us now show that $\text{SPO}_q(2n|2m)$ is a Hopf superalgebra. According to the preceding results we expect that the antipode S of $\text{SPO}_q(2n|2m)$ is given by

$$S(\bar{T}) = \bar{T}' \tag{5.6}$$

(to be interpreted element-wise). Consequently, we have to show that there exists a homomorphism of graded algebras

$$S : \text{SPO}_q(2n|2m) \longrightarrow \text{SPO}_q(2n|2m)^{\text{aop}} \quad (5.7)$$

such that Eqn. (5.6) is satisfied.

To prove this, we work in the graded tensor product

$$\text{Lgr}(V) \overline{\otimes} \text{Lgr}(V) \overline{\otimes} \text{SPO}_q(2n|2m) \simeq \text{Lgr}(V \otimes V) \overline{\otimes} \text{SPO}_q(2n|2m) .$$

In that algebra, we have (with our usual notation)

$$(R \otimes 1) \overline{T}_1 \overline{T}_2 = \overline{T}_2 \overline{T}_1 (R \otimes 1) ,$$

and also

$$\overline{T}_1^{-1} = \overline{T}'_1 \quad , \quad \overline{T}_2^{-1} = \overline{T}'_2$$

(see Eqn. (5.4)). This implies that

$$(R \otimes 1) \overline{T}'_2 \overline{T}'_1 = \overline{T}'_1 \overline{T}'_2 (R \otimes 1) .$$

A moment's thought then shows that this is the RTT-relation (3.4) for \overline{T}' in $\text{SPO}_q(2n|2m)^{\text{aop}}$.

This result implies that there exists a homomorphism of graded algebras

$$S_0 : A(R) \longrightarrow \text{SPO}_q(2n|2m)^{\text{aop}}$$

such that

$$S_0(T) = \overline{T}' .$$

It is easy to check that

$$S_0(Q') = \omega(Q'') = 1 \quad , \quad S_0(Q'') = \omega(Q') = 1 .$$

Consequently, S_0 induces the homomorphism (5.7) we are looking for. It is now easy to see that S is an antipode: The identities to be proved hold (by construction) when evaluated on the generators $\overline{t}_{i,j}$ of $\text{SPO}_q(2n|2m)$, and this implies that they hold on all of $\text{SPO}_q(2n|2m)$.

Summarizing, we have shown that $\text{SPO}_q(2n|2m)$ is a Hopf superalgebra. In the sequel, we shall *simplify the notation* and write $t_{i,j}$ instead of $\overline{t}_{i,j}$. Then the antipode is uniquely fixed by the equation

$$S(T) = T' ,$$

where

$$T' = F_r^{-1} T^{\text{st}} F_r ,$$

or more explicitly

$$t'_{i,j} = \sigma_i \sigma_{i,j} ((C^q)^{-1})_{i,-i} t_{-j,-i} C^q_{-j,j} \quad \text{for all } i, j \in I.$$

We note that

$$S^2(t_{i,j}) = d_i d_{-j} t_{i,j} \quad \text{for all } i, j \in I,$$

where

$$d_i = \sigma_i ((C^q)^{-1})_{i,-i} C^q_{i,-i} \quad \text{for all } i \in I.$$

Obviously, we have

$$d_{-i} = d_i^{-1} \quad \text{for all } i \in I.$$

In $\text{SPO}_q(2n|2m)$, the relations (5.1) and (5.2) take the form

$$\sum_{a,b \in I} \sigma_{\ell,b} \sigma_{b,a} C^q_{a,b} t_{a,k} t_{b,\ell} = C^q_{k,\ell} \quad (5.8)$$

$$\sum_{a,b \in I} \sigma_{i,b} ((C^q)^{-1})_{a,b} t_{i,a} t_{j,b} = \sigma_{i,j} ((C^q)^{-1})_{i,j} \quad (5.9)$$

for all $i, j, k, \ell \in I$. These relations have a simple interpretation. Eqn. (5.8) is equivalent to the fact that the linear form $\tilde{b}^q : V \otimes V \rightarrow \mathbb{K}$ associated to the bilinear form b^q is a homomorphism of graded right $\text{SPO}_q(2n|2m)$ -comodules, and also to the fact that the element

$$a' = \sum_{i,j \in I} \sigma_{j,i} C^q_{i,j} e'_i \otimes e'_j \quad (5.10)$$

is an invariant of the graded left $\text{SPO}_q(2n|2m)$ -comodule $V^{*\text{gr}} \otimes V^{*\text{gr}}$, in the well-known sense that

$$\delta'_2(a') = 1 \otimes a'.$$

Similarly, Eqn. (5.9) is equivalent to the fact that the linear form

$$\tilde{b}'^q : V^{*\text{gr}} \otimes V^{*\text{gr}} \longrightarrow \mathbb{K}$$

defined by

$$\tilde{b}'^q(e'_i \otimes e'_j) = \sigma_{i,j} ((C^q)^{-1})_{i,j} \quad \text{for all } i, j \in I,$$

is a homomorphism of graded left $\text{SPO}_q(2n|2m)$ -comodules, and also to the fact that the element

$$a = \sum_{i,j \in I} ((C^q)^{-1})_{i,j} e_i \otimes e_j \quad (5.11)$$

is an invariant of the graded right $\text{SPO}_q(2n|2m)$ -comodule $V \otimes V$, in the sense that

$$\delta_2(a) = a \otimes 1.$$

Of course, this is exactly what we should guess.

According to Section 2, we expect that the Hopf superalgebra $\text{SPO}_q(2n|2m)$ is coquasitriangular. More precisely, we expect that the universal r -form ϱ on $A(R)$ (as specified in Theorem 1) induces a universal r -form on $\text{SPO}_q(2n|2m)$.

In order to show this we have to prove that ϱ vanishes on $J(Q) \times A(R)$ and on $A(R) \times J(Q)$ (where $J(Q)$ is the ideal of $A(R)$ used in the definition of $\text{SPO}_q(2n|2m)$, see Eqn. (5.5)). The basic ingredients of the proof are the equations

$$\varrho(C_{k,\ell}^q(Q' - 1), t_{i,j}) = 0 \quad , \quad \varrho(t_{i,j}, C_{k,\ell}^q(Q' - 1)) = 0 \quad ,$$

which hold for all $i, j, k, \ell \in I$. These are equivalent to the equations

$$R^{\text{st}_1} \circ (f_r \otimes \mathbb{I}) \circ R = f_r \otimes \mathbb{I} \quad , \quad R^{\text{st}_2} \circ (\mathbb{I} \otimes f_\ell) \circ R = \mathbb{I} \otimes f_\ell \quad ,$$

respectively, which have been proved in Ref. [9] (see Eqn. (7.5) of that reference). We recall that f_ℓ and f_r are the linear maps of V into $V^{*\text{gr}}$ given by

$$f_\ell(e_j) = \sum_{i \in I} C_{j,i}^q e'_i \quad , \quad f_r(e_j) = \sum_{i \in I} \sigma_{j,i} C_{i,j}^q e'_i$$

for all $j \in I$.

According to the preceding remarks, the bilinear form ϱ induces a bilinear form $\bar{\varrho}$ on $\text{SPO}_q(2n|2m)$. Obviously, this form satisfies the Eqns. (2.17)–(2.19). In order to prove that $\bar{\varrho}$ is convolution invertible, one might think we have to show that the inverse ϱ' of ϱ also induces a bilinear form on $\text{SPO}_q(2n|2m)$. However, it is not necessary to prove this independently: Since $\text{SPO}_q(2n|2m)$ is a Hopf superalgebra, it is known (and easy to see) that the bilinear form $\bar{\varrho}'$ on $\text{SPO}_q(2n|2m)$, defined by

$$\bar{\varrho}'(b, b') = \bar{\varrho}(S(b), b') \quad \text{for all } b, b' \in \text{SPO}_q(2n|2m) \quad ,$$

is an inverse of $\bar{\varrho}$.

Simplifying the notation, we have proved the following theorem.

Theorem 2. The Hopf superalgebra $\text{SPO}_q(2n|2m)$ is coquasitriangular. More precisely, there exists a unique universal r -form ϱ on $\text{SPO}_q(2n|2m)$ such that

$$\varrho(t_{ik}, t_{j\ell}) = \sigma(\eta_i - \eta_k, \eta_j - \eta_\ell) R_{ij,k\ell} \quad \text{for all } i, j, k, \ell \in I. \quad (5.12)$$

(We are using the notation introduced in Eqn. (3.1). Since R is homogeneous of degree zero, the factor $\sigma(\eta_i - \eta_k, \eta_j - \eta_\ell)$ can be replaced by $\sigma_i \sigma_k$, and also by $\sigma_j \sigma_\ell$.)

Finally (and once again according to Section 2) we expect that the Hopf superalgebras $U_q(\mathfrak{spo}(2n|2m))$ and $\text{SPO}_q(2n|2m)$ are dual to each other in some sense. More precisely, we expect that there exists a Hopf superalgebra pairing of $\text{SPO}_q(2n|2m)$ and $U_q(\mathfrak{spo}(2n|2m))$. This can be proved, as follows.

Consider the vector module V of $U_q(\mathfrak{spo}(2n|2m))$ (see Ref. [9]), and let π be the graded representation of $U_q(\mathfrak{spo}(2n|2m))$ afforded by it. Define the linear forms $\pi_{i,j} \in U_q(\mathfrak{spo}(2n|2m))^\circ$ by

$$\pi_{i,j}(h) = \langle e'_i, \pi(h)e_j \rangle \quad \text{for all } i, j \in I \text{ and all } h \in U_q(\mathfrak{spo}(2n|2m)).$$

We know that \hat{R} is an endomorphism of the graded $U_q(\mathfrak{spo}(2n|2m))$ -module $V \otimes V$. Defining the elements π_1 and π_2 of the superalgebra

$$\text{Lgr}(V) \overline{\otimes} \text{Lgr}(V) \overline{\otimes} U_q(\mathfrak{spo}(2n|2m))^\circ \simeq \text{Lgr}(V \otimes V) \overline{\otimes} U_q(\mathfrak{spo}(2n|2m))^\circ \quad (5.13)$$

as in Eqn. (2.13), it is easy to see that this is equivalent to the fact that the equation

$$(\hat{R} \otimes \varepsilon) \pi_1 \pi_2 = \pi_1 \pi_2 (\hat{R} \otimes \varepsilon)$$

holds in the algebra (5.13), where ε is the counit of $U_q(\mathfrak{spo}(2n|2m))$, i.e., the unit element of $U_q(\mathfrak{spo}(2n|2m))^\circ$.

By definition of $A(R)$, this implies that there exists a unique algebra homomorphism

$$\chi_0 : A(R) \longrightarrow U_q(\mathfrak{spo}(2n|2m))^\circ$$

such that

$$\chi_0(t_{i,j}) = \pi_{i,j} \quad \text{for all } i, j \in I.$$

Obviously, χ_0 is a bi-superalgebra homomorphism, and we have

$$\chi_0(C_{k,\ell}^q Q') = \sum_{a,b \in I} \sigma_{\ell,b} \sigma_{b,a} C_{a,b}^q \pi_{a,k} \pi_{b,\ell} \quad \text{for all } k, \ell \in I.$$

Since the bilinear form b^q on V is $U_q(\mathfrak{spo}(2n|2m))$ -invariant (equivalently, since the associated linear form \tilde{b}^q on $V \otimes V$ is a homomorphism of graded $U_q(\mathfrak{spo}(2n|2m))$ -modules), the right hand side is equal to $C_{k,\ell}^q \varepsilon$. Consequently, χ_0 induces a bi-superalgebra homomorphism

$$\chi : \text{SPO}_q(2n|2m) \longrightarrow U_q(\mathfrak{spo}(2n|2m))^\circ$$

such that

$$\chi(t_{i,j}) = \pi_{i,j} \quad \text{for all } i, j \in I.$$

Since $\text{SPO}_q(2n|2m)$ and $U_q(\mathfrak{spo}(2n|2m))^\circ$ are Hopf superalgebras, χ is known to be a Hopf superalgebra homomorphism. This implies (indeed, is equivalent to the fact) that the bilinear form

$$\varphi : \text{SPO}_q(2n|2m) \times U_q(\mathfrak{spo}(2n|2m)) \longrightarrow \mathbb{K}$$

defined by

$$\varphi(a, h) = \langle \chi(a), h \rangle \quad \text{for all } a \in \text{SPO}_q(2n|2m) \text{ and all } h \in U_q(\mathfrak{spo}(2n|2m)) \quad (5.14)$$

is a Hopf superalgebra pairing.

Summarizing, we have proved the following theorem.

Theorem 3. There exists a unique Hopf superalgebra pairing

$$\varphi : \text{SPO}_q(2n|2m) \times U_q(\mathfrak{spo}(2n|2m)) \longrightarrow \mathbb{K}$$

such that

$$\varphi(t_{i,j}, h) = \pi_{i,j}(h) \quad \text{for all } i, j \in I \text{ and all } h \in U_q(\mathfrak{spo}(2n|2m)).$$

Let A be the image of $\text{SPO}_q(2n|2m)$ in $U_q(\mathfrak{spo}(2n|2m))^\circ$ under the homomorphism χ . Obviously, A is the subalgebra of $U_q(\mathfrak{spo}(2n|2m))^\circ$ that is generated by the elements $\pi_{i,j}$; $i, j \in I$. Actually, A is a sub-Hopf-superalgebra of $U_q(\mathfrak{spo}(2n|2m))^\circ$ (see the remark below Eqn. (2.7)). By definition, A is dense in $U_q(\mathfrak{spo}(2n|2m))^{\text{gr}}$ if and only if it separates the elements of $U_q(\mathfrak{spo}(2n|2m))$. Using the definition (5.14), we see that the implication

$$\varphi(a, h) = 0 \text{ for all } a \in \text{SPO}_q(2n|2m) \implies h = 0 \quad (5.15)$$

is true for all $h \in U_q(\mathfrak{spo}(2n|2m))$ if and only if A is dense in $U_q(\mathfrak{spo}(2n|2m))^{\text{gr}}$. On the other hand, the implication

$$\varphi(a, h) = 0 \text{ for all } h \in U_q(\mathfrak{spo}(2n|2m)) \implies a = 0 \quad (5.16)$$

is true for all $a \in \text{SPO}_q(2n|2m)$ if and only if χ is injective (and hence induces a Hopf superalgebra isomorphism of $\text{SPO}_q(2n|2m)$ onto A). Unfortunately, at present I am not able to prove or disprove one or even both of the implications above. Actually, there is a simple reason to expect that (5.16) is not correct (see Section 7).

6 Graded comodule algebras over $\text{SPO}_q(2n|2m)$

We keep the notation of the preceding sections, in particular, that of Section 5. In the following, we are going to apply the general results of Section 4 to the case of the Hopf superalgebra $\text{SPO}_q(2n|2m)$.

To begin with, we recall some well-known facts (actually, these are true in the general framework of σ -bialgebras). If W is a graded right $\text{SPO}_q(2n|2m)$ -comodule, the dual pairing of Theorem 3 can be used to convert W into a graded left $U_q(\mathfrak{spo}(2n|2m))$ -module W^{mod} . The procedure is the same as that in Section 2. Let

$$\delta : W \longrightarrow W \otimes \text{SPO}_q(2n|2m)$$

be the structure map of W , and let $h \in U_q(\mathfrak{spo}(2n|2m))$ and $x \in W$ be homogeneous elements, of degrees η and ξ , respectively. If

$$\delta(x) = \sum_s x_s^0 \otimes x_s^1,$$

with $x_s^0 \in W$ and $x_s^1 \in \text{SPO}_q(2n|2m)$, the action of h on x is defined by

$$h \cdot x = \sigma(\eta, \xi) \sum_s x_s^0 \varphi(x_s^1, h).$$

If U is a second graded right $\text{SPO}_q(2n|2m)$ -comodule, and if $f: U \rightarrow W$ is a homomorphism of graded $\text{SPO}_q(2n|2m)$ -comodules, then $f: U^{\text{mod}} \rightarrow W^{\text{mod}}$ is a homomorphism of graded $U_q(\mathfrak{spo}(2n|2m))$ -modules. In particular, any graded $\text{SPO}_q(2n|2m)$ -subcomodule of W is a graded $U_q(\mathfrak{spo}(2n|2m))$ -submodule of W^{mod} . Also, if W is the graded tensor product of the graded right $\text{SPO}_q(2n|2m)$ -comodules W_1, \dots, W_p , then W^{mod} is the graded tensor product of the graded left $U_q(\mathfrak{spo}(2n|2m))$ -modules $W_1^{\text{mod}}, \dots, W_p^{\text{mod}}$. In addition, as the reader will expect, if W is the vector comodule V of $\text{SPO}_q(2n|2m)$, then W^{mod} is the vector module V of $U_q(\mathfrak{spo}(2n|2m))$. Finally, if B is a graded right $\text{SPO}_q(2n|2m)$ -comodule algebra, the construction above converts B into a graded left $U_q(\mathfrak{spo}(2n|2m))$ -module algebra (i.e., the structure maps of the algebra B are homomorphisms of graded $U_q(\mathfrak{spo}(2n|2m))$ -modules).

There is an analogous construction that converts graded left $\text{SPO}_q(2n|2m)$ -comodules into graded right $U_q(\mathfrak{spo}(2n|2m))$ -modules, and its properties are completely analogous to those mentioned above. We leave it to the reader to spell out the details.

After these preliminaries, we are ready to discuss the graded right $\text{SPO}_q(2n|2m)$ -comodule algebras. Guided by Section 4, we should look for graded $\text{SPO}_q(2n|2m)$ -subcomodules of $T(V)$. We restrict our attention to quadratic comodule algebras. Correspondingly, we shall first determine the graded $\text{SPO}_q(2n|2m)$ -subcomodules of

$$T_{\leq 2}(V) = T_0(V) \oplus T_1(V) \oplus T_2(V).$$

In view of the introductory remarks, we can do this in two steps: First, we determine the graded $U_q(\mathfrak{spo}(2n|2m))$ -submodules of $T_{\leq 2}(V)$, then we try to show that the $U_q(\mathfrak{spo}(2n|2m))$ -submodules we have found are, in fact, graded $\text{SPO}_q(2n|2m)$ -subcomodules.

In the following, the $\text{SPO}_q(2n|2m)$ -comodules and $U_q(\mathfrak{spo}(2n|2m))$ -modules will simply be called comodules and modules, respectively.

It is easy to see that every graded submodule of $T_{\leq 2}(V)$ is equal to U or $U \oplus T_1(V)$, where U is a graded submodule of $T_0(V) \oplus T_2(V)$. (Of course, we are not interested in the second case, since it leads to trivial comodule algebras.) Similarly, every graded submodule of $T_0(V) \oplus T_2(V)$ is equal to W or $W \oplus (V \otimes V)_s$, where W is a graded submodule of $T_0(V) \oplus (V \otimes V)_a$ (we are using the notation of Ref. [9]).

Finally, every graded submodule W of $T_0(V) \oplus (V \otimes V)_a$ belongs to one of the following two classes.

1) The module W is equal to W_a or $T_0(V) \oplus W_a$, where W_a is a graded submodule of $(V \otimes V)_a$.

2) We have

$$W = \mathbb{K}(g - c) \oplus (V \otimes V)_a^0,$$

where g is any element of $(V \otimes V)_a$ that does not belong to $(V \otimes V)_a^0$, and where $c \in \mathbb{K} = T_0(V)$ is different from zero.

Our next task consists in checking which of these submodules are, in fact, graded subcomodules of $T(V)$. This is done by means of the following observations. (Unfortunately, in the case $n = m = 1$, I haven't solved this problem completely, see below.)

Obviously, $T_1(V)$ is a graded subcomodule of $T(V)$.

Secondly, we know (see Ref. [9]) that the projector of $V \otimes V$ onto $(V \otimes V)_s$ with kernel $(V \otimes V)_a$ can be written as a polynomial in \hat{R} . Since \hat{R} (and hence the projector) is a homomorphism of graded comodules, this implies that $(V \otimes V)_s$ and $(V \otimes V)_a$ are graded subcomodules.

We know that $\tilde{b}^q : V \otimes V \longrightarrow \mathbb{K}$ is a homomorphism of graded comodules (see Section 5), and that $(V \otimes V)_a^0$ is the kernel of the restriction of \tilde{b}^q onto $(V \otimes V)_a$ (see Ref. [9]). Hence $(V \otimes V)_a^0$ is a graded subcomodule as well.

Remark 6.1. We note that in the case $n = m$ there does not exist a projector of $V \otimes V$ onto $(V \otimes V)_a^0$ that would be a homomorphism of graded comodules. Hence in this case the projector method of Ref. [3] cannot be applied.

In Section 5 we have also noted that the element a (see Eqn. (5.11)) is an invariant of the comodule $V \otimes V$. This shows that $\mathbb{K}a$ is a subcomodule of $V \otimes V$.

In the case $n = m = 1$, we also have to check whether V_4 and \overline{V}_4 are graded subcomodules. Unfortunately, I haven't been able to solve this problem, but I think there are good reasons to conjecture that they are not (see Section 7).

Finally, each of the graded comodules of case 2) above is a graded subcomodule as well. In fact, it is equal to the kernel of the linear form

$$f : T_0(V) \oplus (V \otimes V)_a \longrightarrow \mathbb{K}$$

defined by

$$f(\lambda + u) = \tilde{b}^q(g)\lambda + c\tilde{b}^q(u) \quad \text{for all } \lambda \in T_0(V), u \in (V \otimes V)_a,$$

which is a homomorphism of graded comodules since \tilde{b}^q is such (recall that $\tilde{b}^q(g)$ is not equal to zero).

Since, for each of the graded subcomodules of $T_{\leq 2}(V)$ found above, a basis has been given in Ref. [9], the defining relations of the corresponding graded right $\text{SPO}_q(2n|2m)$ -comodule algebra B_r can immediately be written down.

In the following, we shall concentrate on the case 2) above, but allow for $c = 0$ (i.e., we include the case $W = (V \otimes V)_a$). Note that, for $c \neq 0$, the subcomodule W only depends on the residue class $c^{-1}g + (V \otimes V)_a^0$. We fix our conventions by choosing

$$W = \mathbb{K}(t - q^{2d}c) \oplus (V \otimes V)_a^0 .$$

Here, $t \in (V \otimes V)_a$ is the element specified in Eqn. (5.17) of Ref. [9], and $c \in \mathbb{K}$ is an arbitrary scalar. For convenience, we have included the factor q^{2d} ; recall that $d = n - m$. The corresponding graded right $\text{SPO}_q(2n|2m)$ -comodule algebra $B_r(W)$ will be denoted by $A_q^r(n|m; c)$. For $c = 0$ it might be called the superalgebra of functions on the symplecto-orthogonal quantum superspace, for $c \neq 0$ it is the quantum Weyl superalgebra we wanted to construct (see Remark 6.2).

Using the basis of $(V \otimes V)_a^0$ given in Ref. [9] and the formula for t given there, this comodule superalgebra can be described in terms of generators and relations, as follows.

By definition, $A_q^r(n|m; c)$ is the universal superalgebra, generated by elements x_i ; $i \in I$, which are homogeneous of degree η_i and satisfy the following relations:

The basis elements $a_{i,i}$ with $\eta_i = \bar{1}$ give

$$x_i^2 = 0 \quad \text{for } \eta_i = \bar{1} ,$$

the basis elements $a_{i,j}$ with $i < j$ but $i \neq -j$ give

$$x_i x_j = \sigma_{i,j} q x_j x_i \quad \text{for } i < j, i \neq -j ,$$

the tensors a_j , $2 \leq j \leq r$, give

$$q^{-\sigma_{j-1}} x_{-j+1} x_{j-1} - \sigma_{j-1} q x_{j-1} x_{-j+1} = \sigma_{j-1} \sigma_j x_{-j} x_j - \sigma_{j-1} q q^{-\sigma_j} x_j x_{-j} ,$$

and the tensor $t - q^{2d}c$ gives

$$x_{-1} x_1 - q^2 x_1 x_{-1} = q^{2d}c - (q - q^{-1}) \sum_{i=2}^{n+m} ((C^q)^{-1})_{i,-i} x_i x_{-i} .$$

The last $(r-1) + 1$ relations are equivalent to the system

$$(I) \quad q^{\sigma_i} x_{-i} x_i - \sigma_i q^{-1} x_i x_{-i} = -(q - q^{-1}) \sigma_i C_{i,-i}^q \sum_{j < i} ((C^q)^{-1})_{-j,j} x_{-j} x_j + \sigma_i C_{i,-i}^q q^{2d} c ,$$

which, in turn, can be shown to be equivalent to the system

$$(II) \quad q^{-\sigma_i} x_i x_{-i} - \sigma_i q x_{-i} x_i = (q - q^{-1}) \sigma_i C_{-i,i}^q \sum_{j < i} ((C^q)^{-1})_{j,-j} x_j x_{-j} + \sigma_i C_{-i,i}^q c .$$

Both systems hold for $-r \leq i \leq -1$ (and the summation is over all $j \in I$ such that $j < i$).

Finally, the structure map

$$\delta : A_q^r(n|m; c) \longrightarrow A_q^r(n|m; c) \otimes \overline{\text{SPO}}_q(2n|2m)$$

is fixed by Eqn. (4.7).

It can be shown that the canonical image of the invariant $a \in V \otimes V$ is given by

$$\sum_{j \in I} ((C^q)^{-1})_{j, -j} x_j x_{-j} = (q^d - q^{-d})(q - q^{-1})^{-1} q^d c. \quad (6.1)$$

For $n = m$, this element is equal to zero, as it should (since in this case, we have $a \in (V \otimes V)_a^0$).

Using the braid generator \hat{R} , the defining relations of $A_q^r(n|m; c)$ can be written in the following form. Let $M : A_q^r \otimes A_q^r \rightarrow A_q^r$ denote the multiplication map of $A_q^r(n|m; c)$. Then the defining relations are equivalent to

$$M(\hat{R} - q\mathbb{I} \otimes \mathbb{I})(x_i \otimes x_j) = C_{i,j}^q c \quad \text{for all } i, j \in I. \quad (6.2)$$

In the purely bosonic case, this neat formula is due to Kulish [18]. By multiplying this equation by $((C^q)^{-1})_{i,j}$, summing over i, j , and using some formulae given in Ref. [9], we can easily rederive Eqn. (6.1).

In view of the conciseness of Eqn. (6.2), we might want to use this formula to define the algebra $A_q^r(n|m; c)$. Then we would have to show that the Eqns. (4.7) define on $A_q^r(n|m; c)$ the structure of a graded right $\text{SPO}_q(2n|2m)$ -comodule algebra. This will be the case if we can show that the relations (6.2) stem from a graded subcomodule of $T(V)$, in the way described in Section 4. In order to prove this, we consider the linear map

$$F : T_2(V) \longrightarrow T_0(V) \otimes T_2(V),$$

defined by

$$F(u) = \hat{R}(u) - qu - c\tilde{b}^q(u) \quad \text{for all } u \in T_2(V).$$

Since \hat{R} and \tilde{b}^q are homomorphisms of graded comodules, so is F , and its image is the graded subcomodule of $T(V)$ we are looking for.

We close this section by a series of remarks.

Remark 6.2. Suppose that $\mathbb{K} = \mathbb{C}$ (or, more generally, that \mathbb{K} contains the square roots of all of its elements). Then, for fixed numbers m and n , the graded right $\text{SPO}_q(2n|2m)$ -comodule algebras $A_q^r(n|m; c)$ with $c \neq 0$ are all isomorphic. (This follows by simply rescaling the generators with a suitable overall factor.) Thus, with a slight abuse of language, we then may call this superalgebra *the* $\text{SPO}_q(2n|2m)$ -covariant quantum Weyl superalgebra and denote it by $W_q(n|m)$. Needless to say,

there are really innumerable papers on quantum oscillator (super)algebras of all types, and well-known methods to relate the various versions (see Ref. [19] for an overview). Thus the main contribution of the present work to this field is to present an $\text{SPO}_q(2n|2m)$ -covariant quantum oscillator superalgebra and to provide the mathematical basis of this concept (including the construction of the quantum supergroup $\text{SPO}_q(2n|2m)$ itself).

Remark 6.3. The defining relations of the algebra $A_q^r(n|m; c)$ can be simplified in various ways. For example, choose elements $r_i \in \mathbb{K}$; $i \in I$, such that

$$r_i r_{-i} = ((C^q)^{-1})_{-i,i} \quad \text{for } -r \leq i \leq -1 ,$$

and define

$$x'_i = r_i x_i \quad \text{for all } i \in I .$$

Multiplying the i th relation of the system (I) by $((C^q)^{-1})_{-i,i}$, we obtain the following equivalent system for the generators x'_i :

$$q^{-1} x'_i x'_{-i} - \sigma_i q^{\sigma_i} x'_{-i} x'_i = (q - q^{-1}) \sum_{j < i} x'_{-j} x'_j - q^{2d} c \quad \text{for } -r \leq i \leq -1 .$$

Obviously, for $i \neq -j$ the generators x'_i and x'_j satisfy the same relation as x_i and x_j . On the other hand, the system (II) is more complicated for the generators x'_i . Of course, we could also simplify the system (II), at the price that now the system (I) becomes more complicated.

Remark 6.4. In the case $n = m = 1$, a $U_q(\mathfrak{spo}(2|2))$ -covariant quantum Weyl superalgebra of the type considered in this section has been constructed in Ref. [20]. However, contrary to the claim of the authors, they consider the other version of $U_q(\mathfrak{spo}(2|2))$, namely, the one associated to a basis of the root system consisting of two odd roots. For this version of $U_q(\mathfrak{spo}(2|2))$, I have gone through all the steps of Ref. [9] and of the present work, and finally have found (after adequate adjustments) the quantum Weyl superalgebra of Ref. [20].

Remark 6.5. It should be clear that results similar to those obtained above can also be derived in the case of graded left $\text{SPO}_q(2n|2m)$ -comodule algebras (and I have done that almost completely). Actually, most of the necessary technical tools have been mentioned in the present paper. Thus the interested reader should have no difficulties to carry out the details. He/she might then proceed to study the duality between the two classes of comodule superalgebras thus obtained.

7 Discussion

In the following, it will be useful to include the undeformed case $q = 1$ in our discussion. The corresponding Hopf superalgebra $\text{SPO}_1(2n|2m)$ is easily described. For $q = 1$, we have $R = \text{id}_{V \otimes V}$, hence $A(R)$ is the universal supercommutative

algebra, generated by elements t_{ij} ; $i, j \in I$, which are homogeneous of degree $\eta_j - \eta_i$. The Hopf superalgebra $\text{SPO}_1(2n|2m)$ is defined by requiring in addition the relations (5.8) and (5.9) (with $q = 1$), or their matrix equivalents (see Remark 5.2).

A look at the supergroup $\text{SPO}_1(2n|2m)$ helps to understand how the quantum supergroup $\text{SPO}_q(2n|2m)$ should be interpreted. The Lie group “contained” in $\text{SPO}_1(2n|2m)$ is equal to $\text{SP}(2n) \times \text{O}(2m)$. Correspondingly, the algebra of functions $\text{SPO}_1(2n|2m)$ contains an element $B = B_1$, the Berezinian = superdeterminant, such that $B^2 = 1$, but $B \neq 1$. It is important to recall that the Berezinian is multiplicative. By setting B equal to one, we obtain the algebra of functions on a supergroup, whose associated Lie group is equal to $\text{SP}(2n) \times \text{SO}(2m)$, and which, accordingly, might be called $\text{SPSO}_1(2n|2m)$.

It is tempting to conjecture that something analogous should be true in the deformed case. More precisely, there should exist an element $B_q \in \text{SPO}_q(2n|2m)$, which is group-like, homogeneous of degree zero, central, and satisfies $B_q^2 = 1$ but $B_q \neq 1$. (We might even hope to find a preimage of B_q in $A(R)$ with analogous properties.) The quotient of $\text{SPO}_q(2n|2m)$ modulo the graded Hopf ideal generated by $B_q - 1$ would then be a Hopf superalgebra $\text{SPSO}_q(2n|2m)$. Moreover, we expect that the Hopf superalgebra pairing φ given by Theorem 3 satisfies

$$\varphi(B_q - 1, h) = 0 \quad \text{for all } h \in U_q(\mathfrak{spo}(2n|2m)),$$

which implies that φ induces a Hopf superalgebra pairing φ_0 of $\text{SPSO}_q(2n|2m)$ and $U_q(\mathfrak{spo}(2n|2m))$. In particular, the pairing of Theorem 3 would not satisfy (5.16), and the best we could hope for is that φ_0 would satisfy the implication analogous to (5.16). Similarly, we also expect that the universal r -form ϱ on $\text{SPO}_q(2n|2m)$ described in Theorem 2 induces a universal r -form on $\text{SPSO}_q(2n|2m)$.

The foregoing is closely related to the unsolved problem of whether, in the case $n = m = 1$, the $U_q(\mathfrak{spo}(2|2))$ -submodules V_4 and \overline{V}_4 are $\text{SPO}_q(2|2)$ -subcomodules of $V \otimes V$. It is easy to see that V_4 (resp. \overline{V}_4) is an $\text{SPO}_q(2|2)$ -subcomodule of $V \otimes V$ if and only if $t_{2,-2}^2 = 0$ (resp. $t_{-2,2}^2 = 0$). (The fact that V_4 and \overline{V}_4 are $U_q(\mathfrak{spo}(2|2))$ -submodules of $V \otimes V$ implies that the matrix elements $\pi_{2,-2} = \chi(t_{2,-2})$ and $\pi_{-2,2} = \chi(t_{-2,2})$ satisfy the equation $\pi_{2,-2}^2 = \pi_{-2,2}^2 = 0$ in $U_q(\mathfrak{spo}(2|2))^\circ$.) It can be shown that in $\text{SPO}_q(2|2)$ (even in $A(R)$) we have

$$t_{2,-2}^2 t_{-2,-2}^2 = t_{2,-2}^2 t_{2,2}^2 = t_{-2,-2}^2 t_{-2,2}^2 = t_{2,2}^2 t_{-2,2}^2 = 0.$$

In the case $q = 1$, it is the relation $B_1 = 1$ which implies that $t_{-2,-2}$ and $t_{2,2}$ are invertible and hence that $t_{2,-2}^2 = t_{-2,2}^2 = 0$. Accordingly, we conjecture that V_4 and \overline{V}_4 are $\text{SPSO}_q(2|2)$ -subcomodules but not $\text{SPO}_q(2|2)$ -subcomodules of $V \otimes V$.

The construction of the quantum Berezinian B_q (if it exists) is a difficult problem. One may hope to solve it by use of a suitable Koszul complex. In the undeformed case, the method has been described in Ref. [21], the deformed case has been discussed in Ref. [22]. Note, however, that in the latter reference, the case of Iwahori,

Hecke type R -matrices is considered, whereas here we are in the Birman, Wenzl, Murakami setting.

We close this discussion by some general remarks. In the present work, we have studied the symplecto-orthogonal quantum supergroups $\mathrm{SPO}_q(2n|2m)$ and their graded comodule algebras. Actually, the results of the Sections 3 and 4 have been derived under much more general assumptions. Moreover, also the discussion of the Sections 5 and 6 should be applicable to more general cases. A look into Section 5 shows that only some general properties of the R -matrix have been used. Thus, once the R -matrices of (some version of) the orthosymplectic quantum superalgebras in the vector representation have been calculated, it should be easy to extend the results of the Sections 5 and 6 to these cases.

The reader may have noticed that I have been careful not to use the duality of $\mathrm{SPO}_q(2n|2m)$ and $U_q(\mathfrak{spo}(2n|2m))$ beyond the extent to which it is really established (see Theorem 3). This should help to investigate this duality more carefully.

We have not given a basis of the quantum Weyl superalgebra $A_q^r(n|m; c)$. Using the diamond lemma (see Ref. [23]), it should not be difficult to prove that (with respect to some suitable ordering of the indices) the ordered monomials in the generators x_i form a basis of this algebra; of course, the exponent of x_i should be restricted to $\{0, 1\}$ if x_i is odd.

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